

An Economic Model of Friendship: Homophily, Minorities and Segregation*

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Abstract

We develop a model of friendship formation that sheds light on segregation patterns observed in social and economic networks. Individuals come in different types and have type-dependent benefits from friendships; we examine the properties of a steady-state equilibrium of a matching process of friendship formation. We use the model to understand three empirical patterns of friendship formation: (i) larger groups tend to form more same-type ties and fewer other-type ties than small groups, (ii) larger groups form more ties per capita, and (iii) all groups are biased towards same-type relative to demographics, with the most extreme bias coming from middle-sized groups. We trace each of these empirical observations to specific properties of the theoretical model and highlight the role of choice and chance in generating homophilous behavior. Finally we discuss welfare implications of the model.

Keywords: Networks, Homophily, Segregation, Friendships, Social Networks, Integration, Diversity, Minorities

JEL Classification Numbers: D85, A14, J15, J16

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1 Introduction

The network structure of social interactions influences a variety of behaviors and economic outcomes, including the formation of opinions, decisions of which products to buy, investment in education, access to jobs, and social mobility, just to name a few. In particular, the extent to which a society is segregated across different groups can be critical in determining things like how quickly information diffuses, and the extent to which there is under-investment in human capital, among other things. In this paper we examine a fundamental and pervasive phenomenon of social networks which is known as “homophily”.¹ This refers to a tendency of various types of individuals to associate with others who are similar to themselves. Homophily is a term coined by Lazarsfeld and Merton (1954) and it applies very broadly, as measured by age, race, gender, religion, profession and is generally a quite strong and robust observation (see McPherson, Smith-Lovin and Cook (2001) for an overview of research on homophily). Extensive empirical research shows strong evidence of homophily, regardless of along which dimensions types are measured. Given the importance of interaction patterns, developing models that help us to understand homophily is essential.

In this paper we begin by examining some detailed observations about patterns of homophily. Some background comes from an extensive empirical literature in sociology. Beyond that background, we identify three specific patterns of homophily relative to the ‘Adolescent Health’ data set (described in detail below), which examines friendship patterns in a representative sample of U.S. high schools.² We then build a model of friendship formation and show that the model can generate the observed patterns of homophily, and in particular, we trace different aspects of observed homophily to different aspects of the model.

The three main observations that we point out from the data are summarized as follows (and see below for details). Consider a high school and the patterns of friendship within it. In particular, let the type of an individual be their race and let us define an index of homophily to be applied to a given race within a given high school. We keep track of the percentage of each individual’s friendships that are with other individuals of the same race. Then, averaging this fraction across all individuals of a given race provides a measure of how “homophilous” that given race within that given high school is. If race was not a

¹The etymology of the term is simple: homo = self and philia = love.

²Add Health is a program project designed by J. Richard Udry, Peter S. Bearman, and Kathleen Mullan Harris, and funded by a grant P01-HD31921 from the National Institute of Child Health and Human Development, with cooperative funding from 17 other agencies. Persons interested in obtaining data files from Add Health should contact Add Health, Carolina Population Center, 123 W. Franklin Street, Chapel Hill, NC 27516-2524 (addhealth@unc.edu).

factor in the formation of friendships, then cross-race friendships would occur roughly in proportion to the relative preponderance of the races. For instance, in a school that is two thirds of one race and one third of another, if race was not a factor then the more populous race should have roughly two thirds of its friendships with same race, and so its homophily index should be roughly two thirds. A measured homophily index significantly different from two thirds provides evidence that race somehow (possibly indirectly) matters in the formation of friendships. The first main empirical observation is a very basic one that has been documented before, which we refer to as relative homophily: larger groups (measured as a fraction of the population of their respective schools) exhibit higher homophily indices. One would expect this even if race does not matter at all in the formation of friendships. In particular, larger groups form on average more friendships with their own group than smaller groups, and on average form fewer friendships with other groups. The second main empirical observation that we document is that larger groups form significantly more friendships per capita. More specifically, we see that members of a group that comprise a small minority in a school form roughly 5 friendships per capita, while members of groups that comprise large majorities (close to one hundred percent of a school) form on average more than 8 friendships. This pattern is something that we argue must be attributed to race mattering, and in particular we will argue that it traces back to races entering through preferences. The third main empirical observation that we document is that groups tend to have homophily indices that are at least as large as their respective fraction of the population. That is, groups tend to be “inbred” (a term from the literature on homophily) in friendship patterns. Here the new observation is that this inbreeding is essentially absent for groups that form very small or very large fractions of their school, but the inbreeding is very large for groups that form a *middle-ranged* fraction of their school.

With these three observations in hand, we develop a model of friendship formation. In the model, individuals come into a society and form friendships when they enter. Friends are met through a process of random search, and each search entails a fixed cost. The longer an individual spends in the matching process, the more friendships are formed. So, an individual’s decision problem is how many friendships to form, or equivalently, how long to spend in the matching process. There are diminishing returns to forming friendships, and so eventually an individual will exit the process. We study a steady-state equilibrium of this process, where there are inflows and outflows of agents, and these balance. The critical determinant of an individual’s strategy, and thus how long agents of different types stay in the matching process, is his or her preferences and the mix of types that he or she faces. We examine preferences such that an individual could potentially care about the number of

same-type friends that he or she has, as well as the number of different-type friends that he or she has. Once preferences are specified, we have a well-defined system and we show that a steady-state equilibrium always exists and we study its properties. Most importantly, the strategies of the agents, as well as the makeup of the inflowing populations, determine what the matching process looks like. For example, if one group forms $2/3$ of the inflowing population, but stays in the matching twice as long as each of the other groups, then they will form $4/5$ of the stock in the matching process, rather than $2/3$. This comes from examining how many people will be in the matching process at a typical date: two generations of the larger group for every one set of the smaller group, and so the overall proportion of the larger group is $\frac{2 \times 2/3}{2 \times 2/3 + 1 \times 1/3}$. Thus, strategies affect the matching process, which feeds back to affect strategies, and we close this by examining a steady-state equilibrium.

A rough summary of the implications of the model are as follows. The relative homophily observations, that larger groups have a higher absolute homophily, are implied by equilibrium conditions and diminishing returns to friendships, without any further conditions. The requirement that friendships add up across groups implies that larger groups must have smaller number of different-group friendships per capita. Diminishing returns then imply that the smaller group could not be forming more same-type friendships, or it would have more friendships of both types which implies a lower marginal return from spending time in the matching, which could not be consistent with both types following optimal strategies. So, larger groups form more same-type friendships and fewer different-type friendships than smaller groups, which then implies that larger groups have higher homophily indices, as observed in the data. While equilibrium implies relative homophily, it does not imply anything about overall numbers of friends or about inbreeding homophily. We then trace the overall number of friendships formed to preference conditions. If agents have a bias in their preferences, so that they receive higher marginal returns when forming a mix of friendships that is biased towards same-type, then groups that face a matching with more of their same-type will form more total friendships. This effect then feeds back, so that the larger group stays in the matching longer, and so the stocks of agents are further biased towards larger groups. This results in inbreeding homophily, where the larger groups end up forming a percentage of same-type friendships that is larger than their relative presence in the overall population. While the preference bias ends up being consistent with the observation of larger groups forming more friendships per capita, and generates inbreeding homophily for the largest group, the overall patterns of homophily generated across groups do not match the observations. Finally, by considering variations on the matching process itself, inbreeding homophily patterns across the population that match the observed patterns (where middle-sized groups

are the most inbred), are found by allowing groups to match so that they meet same-types at a higher rate than in terms of relative stocks. This has differential effects across population sizes.

We conclude the analysis with an examination of welfare issues. While the model is too stark to take seriously for policy implications, the welfare analysis points out that average welfare depends in sensitive ways to the structure of preferences and matching. This suggests that it is vital to build richer models of friendship formation in order to better understand homophily and self-segregation within schools and to develop well-founded policies in response.

Choice and Chance in Network Formation

Regarding contributions to the literature, to our knowledge, this is the first model of its kind, explicitly modeling friendship formation when homophily is a potential issue. Researchers are well aware that homophily patterns will be governed by both the meeting process and preferences. For example, Moody (2001, page 680) notes that: “friendship segregation results from the multilevel influences of mixing opportunity and individual preference”. However, our model seems to be the first systematic investigation of the different roles of choice and chance in determining the emerging patterns of social ties.³ In terms of the specific observations that we examine, the contributions are as follows. The observation regarding relative homophily, and pointing out that this essentially follows having friendships balance across types, is something that has roots in what is known as “contact theory.” For example, Blau (1977) points out that since each cross-group friendship must involve a member from each group, then smaller groups must have more cross-group friendships on a per capita basis. Overall relative homophily, however, is not implied by this alone, and also requires that preferences have diminishing returns. The second observation that larger groups form more relationships is an empirical finding that has been noted before, and for instance by Marsden (1987) in a study of advice networks. The finding that this is implied by a particular bias in preferences (and will not hold in the absence of such a bias) is new. The third observation that inbreeding has a specific pattern, and is maximal for middle-sized groups also appears to be new, as does the explanation that this must be tied back to the technology of matching. Moody (2001) suggests that when two groups are close to a majority their tendency to inbreed increases, but because of competing groups concerned

³In fact, even more generally, most models of network formation fall either into a class of random graph models governed by chance, or a game theoretic model governed by choice. See Jackson (2006) for some discussion and background.

about sociological factors, which are absent from our preference-based model. Finally, the welfare analysis that accompanies the model is new as well.

We begin the paper with some brief introductory background on homophily before presenting the homophily indices, observations on the Add Health data, the model, and the analysis.

2 Homophily: Some Background and Definitions

2.1 Empirical Background

Homophily has been noted throughout history, and is seen in such adages as “birds of a feather, flock together.” which dates to at least the fifteenth century.⁴ For example, based on a national survey Marsden (1987, 1988) finds that only 8 percent of people have *any* people of another race with whom they “discuss important matters.” More generally, this tendency of similar agents to tie together has been documented by a large sociological literature that has examined many dimensions of diversity.

Explanations for homophily have referred to the different opportunities faced by the member of a referral group to tie with other members of the same group or with members of other groups, as well as to preferences. In his influential paper, Blau (1977) pointed to the role of relative group sizes in determining the opportunities of members of various groups to tie together. Looking at the simple case of two groups, Blau observes that, due to the reciprocal nature of ties, cross-type ties will have a larger weight on the smaller group than on the larger group. This leads to a basic form of homophily where larger groups have fewer different-type friendships relative to their size, so that if all agents form the same number of friendships overall, larger groups would have higher homophily than smaller groups.

As an example, Table 1 summarizes information about friendship links by ethnicity of students in a Dutch high school collected by Baerveldt, Van Duijn, Vermeij and Van Hemert (2004).

Table 1 records the percentage of friendships by race. The table should be read as follows: the cell ij indicates the percentage of j 's friendships that are directed towards race i . Each column therefore sums to 100 percent. If friendships were formed purely at random (as in Blau's analysis), the demographics would lead each group i to receive a share of other

⁴Lazarsfeld and Merton (1954) attribute this to Robert Burton in 1621, but a version of it appears in Philemon Holland's (1601) translation of “Livy's Roman history,” and seems to come from folklore before that.

Percent of Friends by Ethnicity:	Ethnicity of Students				
	Dutch	Moroccan	Turkish	Surinamese	Others
	n=850 %=64	n=62 %=5	n= 75 %=6	n=100 %=8	n=230 %=17
Dutch	79	24	11	21	47
Moroccan	2	27	8	4	5
Turkish	2	19	59	8	6
Surinamese	3	8	8	44	12
Others	13	22	14	23	30

Table 1: Percentage of Links Across Ethnicities in a Dutch High School; from Baerveldt et al (2004).

groups’ friendships which equals i ’s relative size in population. So we would observe a first row of all 64%, a second of all 5%, and so forth. In fact what we see goes beyond this pure effect of demographics: the percentages on the diagonals exceed the relative fractions of the population, and hint a self-bias of groups in the process of friends formation.⁵ Moreover, this tendency to link to own type beyond the effect of demographics is not uniform across groups, and may even offset the effect of demographics on the outwardness of groups. In the case of Table 1, Turkish (a minority group) show a stronger tendency towards intra-group relationships than Dutch (the majority group), after correcting for the effect of opportunity of cross ties (see Baerveldt et al (2004)).

An extensive literature has documented linking tendencies and their relation with other aspects of groups’ configuration. For example, there is some evidence that the percentage of inter-group links can vary nonlinearly and non monotonically with overall measures of heterogeneity in the population (e.g., see Blalock (1967) and Moody (2001)). There have been various theories proposed for this, for instance based on contact theory (e.g., Allport (1954), Blau (1977)), or on power differences and competition (e.g., Giles and Evans (1986)).

In Section 3 we present in some detail the main patterns of homophily that come out of the high school friendships data from the National Longitudinal Study of Adolescent Health. Before that, we define some indexes of homophily.

⁵The “others” category is an exception, but it is a conglomerate of a number of ethnicities.

2.2 Defining Homophily

We propose definitions of homophily that allow for multiple groups and for varying total numbers of ties.

Let N_i denote the number of type i individuals in the population, and let $w_i = \frac{N_i}{N}$ be the relative fraction of type i in the population, where $N = \sum_k N_k$. Let $N_{-i} = \sum_{k \neq i} N_k$ denote the total population of types other than i .

Let s_i denote the average number of friendships that agents of type i have with agents who are of the same type, and let d_i be the average number of friendships that type i agents form with agents of types different than i .

The first index is the obvious one which is a basic homophily index H_i , measuring the average fraction of the ties of individuals of a given type that are with that same type.

DEFINITION 1 *The homophily index H_i is given by*

$$H_i = \frac{s_i}{s_i + d_i}.$$

The index H_i measures the relative frequencies of same-type versus different-type friendships. As observed by Blau (1977), this ratio will partly be determined by opportunities, reflecting the relative sizes of groups' populations. This aspect of homophily is captured by the following definition.

DEFINITION 2 *The profile of friendships $(s, d) = (s_1, d_1, s_2, d_2, \dots, s_K, d_K)$ satisfies **baseline homophily** if for all i :*

$$H_i = w_i.$$

If there are two groups, then Blau's observation that cross-type friendships require one from each type implies that $N_1 d_1 = N_2 d_2$, so that $N_1 > N_2$ implies $d_1 < d_2$. In other words, larger groups *must* have fewer ties with other types per capita, based solely on simple accounting. If in addition $s_1 > s_2$, so that larger groups have more same-type friendships per capita, then we have a pattern that we call relative homophily.

DEFINITION 3 *A profile $(s, d) = (s_1, d_1, s_2, d_2, \dots, s_K, d_K)$ satisfies **relative homophily** if $N_i > N_j$ implies $s_i > s_j$ and $d_i < d_j$, and then $H_i > H_j$.*

Relative homophily is satisfied in the case of baseline homophily where all friendships are formed in relative population proportions.

However, beyond relative homophily, friendships can be biased with respect to baseline homophily, as we shall see in the data to follow. This suggests that raw demographics are not the only source of homophily. The observed tendency of friendships to be biased towards own types beyond the effect of relative population sizes has been referred to in the sociological literature as "inbreeding homophily" (see e.g. Coleman (1958), Marsden (1987), McPherson et al (2001)).

DEFINITION 4 *The profile (s, d) satisfies **inbreeding homophily** for type i if:*

$$H_i > w_i.$$

Clearly, inbreeding homophily could be due to aspects of choice, but might also be due to opportunities that are not reflected in the relative sizes of groups, and that may stem from internal characteristics of the organization in which ties are formed (e.g., tracking, classes and extracurricular activities in schools).

One can also have the reverse condition where friendships are biased towards having cross-type friendships.

DEFINITION 5 *The profile (s, d) satisfies **heterophily** for type i if:*

$$H_i < w_i.$$

Generally, there is a difficulty in simply measuring homophily according to H_i . For example, consider a group that is 95 percent of the population. Suppose that its same-type friendships are 99 percent of its friendships. Compare this to a group that forms 50 percent of a population and has 54 percent of its friendships being same-type. Although both have a bias of an extra 4 percent of friendships relative to its size in the population, the first group is more biased in terms of what fraction of those ties are same-type relative to the makeup of ties that it could have formed. In order to take care of this, use the following index developed by Currarini, Jackson and Pin (2007) which normalizes the homophily index by the potential extent to which a group could be biased.⁶

DEFINITION 6 *The **inbreeding homophily** of type i is*

$$IH_i = \frac{\frac{s_i}{s_i+d_i} - w_i}{1 - w_i} = \frac{H_i - w_i}{1 - w_i}.$$

⁶As will be clear from the data, this also corrects an issue that the raw homophily index is heteroschedastic when viewed relative to group fraction w_i , while this normalized index has a more constant variance as a function of w_i .

This index measure the amount of bias with respect to baseline homophily as it relates to the maximum possible bias (the term $1 - w_i$). It can be easily checked that we have inbreeding homophily for type i if and only if $IH_i > 0$, and inbreeding heterophily for type i if and only if $IH_i < 0$. The index IH_i measures the amount of bias with respect to baseline homophily divided by the maximum potential bias available to type i . In fact, pure baseline homophily would imply that $\frac{s_i}{s_i+d_i} = w_i$, while the upper bound for the term $\frac{s_i}{s_i+d_i}$ is 1, from which the denominator of the expression for IH_i . Thus, the measure of inbreeding homophily is 0 if there is pure baseline homophily, and 1 if a group completely inbreeds.⁷ Measuring inbreeding homophily as a share of the maximum potential homophily adjusts for group size to keep track of the amount of inbreeding homophily that a group could potentially display.

3 Details on the Structure of Friendships and Homophily

Table 2 presents friendships in an American High School, from the Add Health data set.⁸

Table 2 reads as Table 1. The numbers on the main diagonal record the friendships that are of same-type. These values are substantially higher than the relative group’s size for the two larger groups, and lower for the smallest group (Hispanics), reflecting inbreeding homophily for both whites and blacks, and some heterophily for Hispanics. The IH index of inbreeding homophily is 0.69 for whites (whose relative population is 51%), 0.76 for blacks (relative population 38%) and -0.11 for Hispanics (2% of population). This non-monotonic trend of the IH index will be discussed in detail in Sections 6 and 7 of this paper.

The network of friendships by race in this high school is pictured in Figure 1.

⁷One could also define a heterophily index, which would be $\frac{\frac{s_i}{s_i+d_i} - w_i}{-w_i}$, reflecting the extent to which a group is outgoing. It would be 0 at baseline homophily and 1 if a group only formed different-type friendships.

⁸The data from National Longitudinal Survey of Adolescent Health (commonly referred to as “Add Health”) were collected over several years starting in 1994 over a carefully selected sample of high schools and middle schools. There are 112 schools in the data set with behavioral and demographic data, and here the data are from 84 schools for which extensive network information was obtained. The data are based on student interviews. The friendship data were based on reports of friendships by each student. Students were shown a list of all the other students in the school and permitted to name up to five friends of each sex. Only 3 percent nominated ten friends, and only 24 percent hit the constraint on one of the sexes, and so the constraints do not seem to be a substantial measurement issue (see Moody (2001) for more discussion). The data include information about how much interaction there is between individuals, which we do not make use of as it does not add much to our analysis. Here a tie is present if either student mentioned the other as a friend. Students could also identify other students with whom they had sentimental relations, which are not reported among friendships. The attribution of race is based on the self-reported classification.

Percent of Friends by Ethnicity:	Ethnicity of Students			
	White	Black	Hispanic	Others
	n=131 %=51	n=96 %=38	n= 13 %=5	n=15 %=6
White	85	7	47	74
Black	4	85	46	11
Hispanic	4	6	2	4
Others	7	2	5	11

Table 2: Percentage of Links Across Ethnicities in an American School; from Add Health 1994 Data.

Figure 1 shows the stark tendency of white and black students to form distinct communities (high levels of inbreeding homophily), and the different behavior of Hispanic students (less numerous), who integrate more with other races and fail to form an independent community.

We now turn to examining the three observations mentioned in the introduction. These observations are from Currarini, Jackson and Pin (2007), where we provide additional detail in the analysis including relating homophily to other observables.

3.1 Relative Homophily

In each of the 84 high schools there are 4 different racial characterizations, leading to 336 potential observations. There are 31 cases where there are no students of a given race, and so we have 305 total observations. Each observation is then a race within a particular school.

Figure 2 shows how relative homophily is pervasive in the Add Health schools. Each dot represents s_i (left part) and d_i (right part) of a particular racial group i in one of the 84 Add Health high schools. In the x -axis we have the percentage w_i of a group i in its school. The plots show the increase in the average number s_i of same type friends (left frame) and decrease the average number d_i of different-type friends (right frame), as w_i varies from 0 to 100%.⁹

If s_i increases and d_i decreases in w_i , then $\frac{s_i}{s_i+d_i}$ necessarily increases in w_i . Figure 3 illustrates the observed distribution of the homophily index $H_i = \frac{s_i}{s_i+d_i}$ as a function of w_i . The 45 degree line provides the baseline homophily benchmark.

⁹The slopes are 7.6 with a t of 32, and a slope of -5.7 with a t of -15.6, respectively.

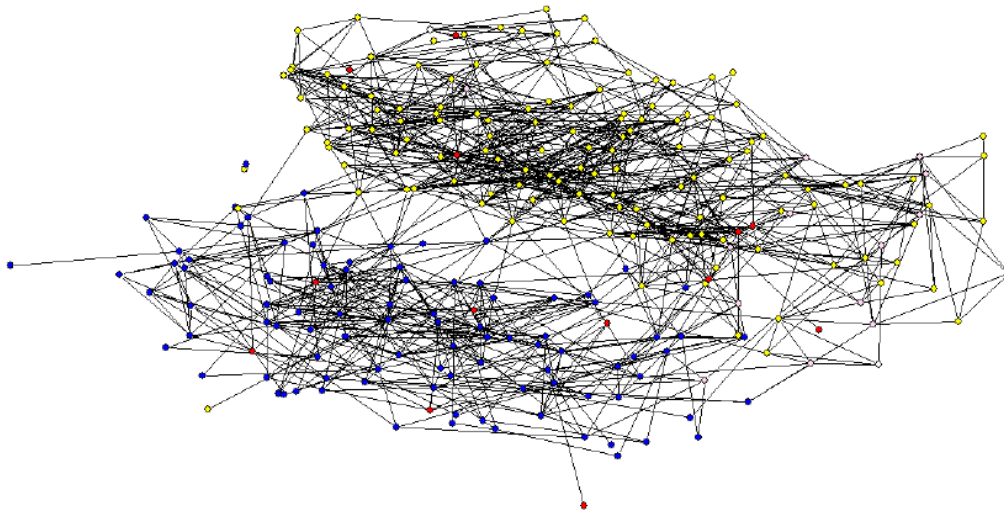


Figure 1: Friendships network in a U.S. School. Colors identify races: Yellow=Whites; Blue=Blacks; Red=Hispanic.

Beyond the clear positive tendency of increasing homophily with group size (as a fraction of population),¹⁰ we also note that the baseline homophily line acts as a lower envelop of the observed dots, which is evidence of pervasive inbreeding homophily that we examine in more detail in Section 3.3.¹¹

3.2 Group Size and the Total Number of Friends

The observed inbreeding homophily could be coming from a variety of sources. Various groups may end up spending more time with their own types just as a result of a segregated organization within the school, or because of opportunities that differ from those that are described by a purely random process of friendship formation on the overall school population. Academic tracking is one example where correlation may be present, inducing segregation even within an heterogeneous school (see Feld, 1981). We may therefore end up observing a fair amount of inbreeding homophily even without any preference or choice-based reason for it.

¹⁰The slope of a regression is .98 with a t -statistic of 31. We fit a nonlinear curve below.

¹¹Figure 3 is very similar to Figure 6 in Echenique and Fryer (2007) who are interested in defining segregation measures and who develop a measure based on the spectral decomposition of the friendship matrix and test it on the same data. Thus, the same pattern exists for other measures as well.

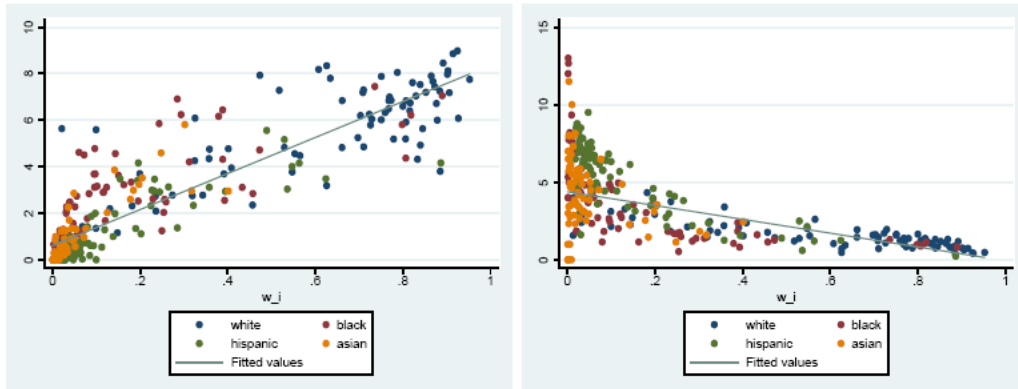


Figure 2: Relative homophily. s_i (left) and d_i (right), varying the percentage w_i of a group in the school.

We now present a critical observation that provides evidence that preferences do play an important role in the racial mix of friendships. This is something that we also investigate in detail with respect to the model. This concerns observations relating groups' sizes to the total number of friendships. This relation cannot be generated solely by the pure effect of opportunities, as without any preference bias. One would have to tell a story explaining why opportunities are more numerous as a function of group size, when group size does not have any preference component (and independent of race).

In Figure 4, there is a clear positive relation between the size of racial groups and the total number of per capita friends that they form. Together with a pervasive presence of inbreeding homophily, this positive relation seems to be a significant result from the Add Health survey data. It is notable that this aspect of friendship formation was neglected in Blau's (1977) analysis of opportunity driven homophily, and has not been accounted for as a possible source of differences in the observed behavior of difference racial groups.^{12, 13}

¹²Marsden (1987) observes a similar pattern in his empirical study of the "discussion network" based on a national U.S. survey, but does not discuss a connection between such a pattern and other aspects of friendship formation.

¹³The coefficient of the regressed line is 3.3, with a t-statistic of 7.1, and the intercept is 5.0 with a t-statistic of 29.

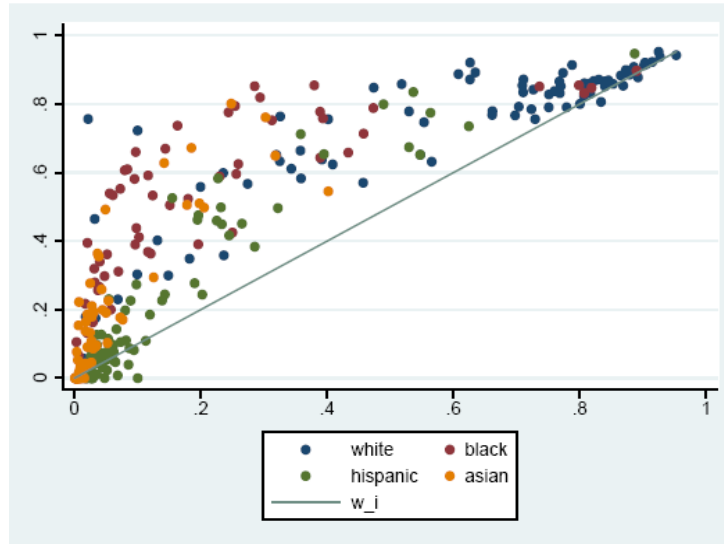


Figure 3: The homophily index.

3.3 Inbreeding Homophily

As mentioned above, another feature of the data is that there is inbreeding homophily for most groups. Moreover, there is a significant and distinctive pattern to IH_i as a function of relative group size. Figure 5 illustrates inbreeding homophily as a function of group size (and also as a function of race).¹⁴

Figure 5 indicates two clear patterns. First, there is inbreeding homophily for most groups, with some exception for the smallest groups. Second, there is a clear bell shaped curve where the measure is close to 0 at the extremes and very high (reaching almost 80 percent) near the middle.

4 A Search-Based Model of Links among Diverse Types

We now examine a search/matching-based model of friendship formation. Such search/matching models have been used in a variety of contexts (e.g., labor markets as in Mortensen and Pissarides (1994)). The model introduced here is distinct in that agents form many friendships.

¹⁴In a regression of IH_i versus w_i and w_i^2 , we find a coefficient of 2.2 on w_i and a coefficient of -2.3 on w_i^2 , with t-statistics of 17 and -16, respectively. The intercept term is .1 and insignificant. Running a regression of the inbreeding homophily index on the variable $w_i(1 - w_i)$ and forcing a zero intercept, we obtain a coefficient of 2 (standard error 0.07), with a t-value of 27. In Currarini, Jackson and Pin (2007) we analyze these data in more detail.

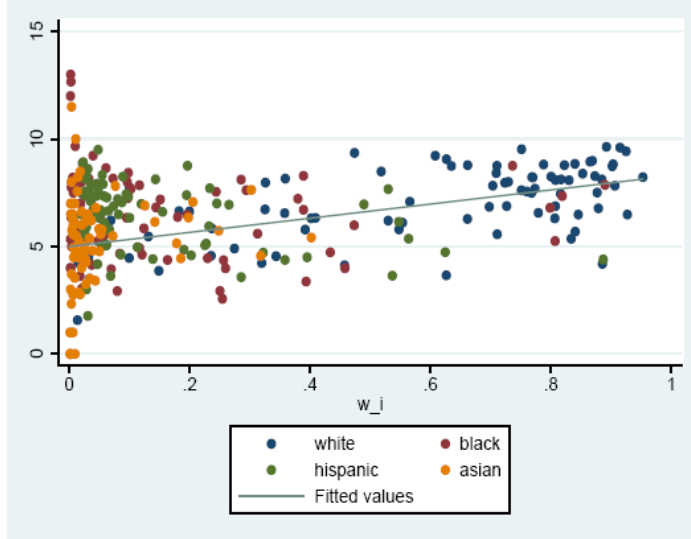


Figure 4: Larger groups make more friends

4.1 Agents, Types and Payoffs

Agents come in a finite variety of types $T = \{1, \dots, K\}$. These might correspond to ethnicities, religious affiliations, professions, age, etc., or some combinations of traits.

The utility of an agent depends on the numbers of his or her friends¹⁵ who are of his or her same type and who are of different types. For the purposes of this model, the agent only distinguishes between “same” and “different,” and does not distinguish in any finer way among differences. This is consistent with empirical evidence. For example Marsden (1988) does not find any significant distinction between friendships and races after accounting for homophily. As McPherson et al (2001) point out, the “key distinction appears to be same–different”.

The total utility to an agent i who has s_i same-type friends and d_i different-type friends is $U(s_i, d_i)$.

Note that the function U is not indexed by the type of the agent. Thus agents’ base preferences are the same, but their resulting outcomes may differ depending on the society that they face.

For convenience, we allow agents to form fractional friendships. This allows us to treat U as strictly increasing, continuous, with continuous first- and second-order partial deriva-

¹⁵The word “friend” is used throughout, but this might also correspond to some other sort of relationship, depending on the application.

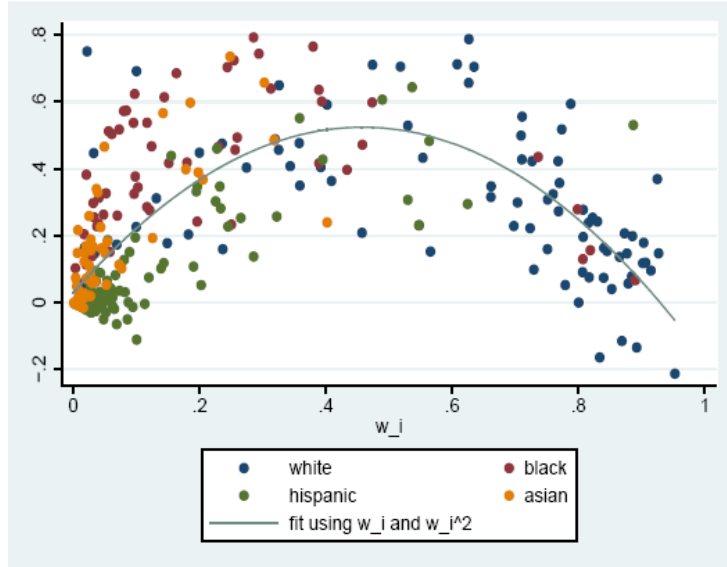


Figure 5: Patterns of Inbreeding Homophily by Relative Group Size and Race.

tives.¹⁶ Let $U_s(s_i, d_i)$ denote the partial derivative of U with respect to s evaluated at (s_i, d_i) , and use similarly standard notation for other partial derivatives. U is strictly concave in s_i and is strictly concave in d_i .

We also maintain an assumption that the utility function U exhibits *overall diminishing returns to friendships*. That is,

$$U(as, ad) < aU(s, d),$$

for all (s, d) and $a > 1$. Diminishing returns is simply a standard decreasing returns to scale condition, here applied to preferences for friendship. Geometrically, it requires that the marginal utility of friendships decreases along any ray going out of the origin in the space (s, d) . This property is weaker than overall concavity of U .¹⁷ This condition is equivalent to the following condition on the total derivative

$$sU_s(as, ad) + dU_d(as, ad) < sU_s(s, d) + dU_d(s, d) \quad (1)$$

for all (s, d) and $a > 1$.

Diminishing returns to friendships is satisfied if s and d are substitutes in agents' preferences, given the strict concavity of U in both s and d . It is also satisfied when s and d are

¹⁶The case in which agents have satiated preferences is also of interest in various applications and is treated in the Supplementary Material to this paper.

¹⁷It implies concavity of U if U is quasiconcave and homothetic (see Friedman, 1973).

complements, as, for instance, in Cobb-Douglas utility functions in which the sum of powers is less than one.

Finally, to ensure the existence of optimal plans, we assume that marginal utilities vanish as s and d go to infinity. More precisely, for all $\varepsilon > 0$ there exist (s', d') such that $U_s(s, d) < \varepsilon$ and $U_d(s, d) < \varepsilon$ for all $s > s'$ and $d > d'$.

4.2 Examples

Before moving on, let us simply mention a few examples to keep in mind that suggest preferences on same/different types of friends.¹⁸

- **Information Networks** Consider a situation where individuals receive information from their friends. In particular, the information they learn from a same-type individual is more likely to be correlated with their own information, and that of different types is more likely to be uncorrelated. However, a same-type individual is easier to communicate with.
- **Professional Networks** Here, same-type individuals are easy to communicate with, but offer less creative synergy.
- **Purely Social Networks** This is a situation where one shares more interests with same-type individuals.
- **Risk Sharing Networks** Here, a same-type individual's income could be more highly correlated with own income, which offers less opportunity for risk-sharing. However, they could be closer geographically or socially, and thus easier to arrange a viable (and self-enforcing) risk-sharing agreement with.

4.3 A Matching Process

We examine a discrete time matching process with a new inflow of agents of mass N_i of type i in each period. This mass embodies a continuum of agents, which is convenient for identifying steady-states of the matching process, as we outline in detail in Appendix A. The agents choose how many periods to stay in the matching, and form friendships in

¹⁸For a more detailed analysis of the characterization and properties of social networks see the survey by Jackson (2006), a model of network formation that originates segregation has been recently proposed by Watts (2006), while recent developments are available in Ioannides and Datcher Loury (2004) and Page (2007) for the professional networks, and in Bramoullé and Kranton (2007) for the risk sharing networks.

each period that they are in the matching. Then they exit. Given stocks of agents in the matching, an agent will end up meeting a same-type with a probability proportional to the stock of same-types, and a different-type with the remaining probability.

In the analysis of the matching model we only consider strategies in which agents always accept any match. In a steady state where matching probabilities are constant in time, it is easy to see that non satiated preferences and the assumption of a continuum of agents directly imply that such strategies are dominant (remember that the cost of search is paid before the realization of each random match). In this case, the main decision for an agent is how long to stay in the matching process, which ultimately determines how many friends they form.

For any given agent of type i who searches for a new friend, the agent meets another agent of type i with probability q_i and an agent of another type with a probability $1 - q_i$. These probabilities will be endogenous in equilibrium.

If an agent of type i stays in the matching for a time t_i , then he or she forms a total of t_i friends, with a proportion $q_i t_i$ of them being of same type, and $(1 - q_i)t_i$ being of different type. Again, this presumes that agents form friends with anyone they meet in the matching process, which is without loss of generality given the preference structure.

In Appendix A we provide a detailed derivation of a random matching process with a continuum of agents allowing for fractions of matches, and show how taking limits as time between matchings goes to 0 can justify exactly our formulation as the limit. In particular, we are implicitly assuming a sort of Law of Large Numbers here, which presents some technical challenges in matching processes with continua of agents. Recent work by Alós-Ferrer (1999) and Duffie and Sun (2007) provides a foundation on which we build to provide a complete justification for the assumed matching system.

4.4 Costs

For each unit of matching the agent incurs a cost $c > 0$. In this model, adding discounting only complicates the expressions without adding any insight given direct costs to matching, and so we omit discounting.

The fact that agents accept any friend who is met implies that we can keep track of costs

as a function of t_i , so that costs are¹⁹

$$C(t_i) = ct_i.$$

4.5 The Friendship Formation Decision Problem

Given the matching probability q_i for each type i , we have a well-defined process for each agent. In a steady state where q_i is constant, agents solve the following problem:

$$\max_{t_i} U(q_i t_i, (1 - q_i) t_i) - ct_i. \quad (2)$$

The following lemma states a straightforward but useful necessary condition for an optimum.

LEMMA 1 *An optimal choice of the time spent in the matching t_i for an agent of type i given matching probabilities q_i and $1 - q_i$ satisfies the following condition:*

$$q_i U_s(q_i t_i, (1 - q_i) t_i) + (1 - q_i) U_d(q_i t_i, (1 - q_i) t_i) = c. \quad (3)$$

5 Endogenous Matching and Steady State Equilibrium

We now define an overall equilibrium notion that accounts for the relation between players' strategies and the type of matching each player faces in the system. We need to keep track of the in-flows of agents into the society, as well as the out-flows of those who have finished searching, and the stock of agents at different stages of their search process.

Consider flows of agents into the economy such that at each time there is a new measure N_i of type i coming in. Take these to be continua, so that the matching has precise numbers of matches in each period. These inflows, together with a specification of strategies leads well-defined measures of agents at different phases of their matching process at any given time, and we can define corresponding steady-state q_i 's.

A steady-state equilibrium of the system for a given set of N_i 's, is thus a specification of strategies for each type such that the resulting stocks of agents at each point in their search process lead to the q_i 's that justify the strategies, and such that the outflow of agents is the same as the inflow.

Before presenting the definitions, we present an example that illustrates the ideas.

¹⁹We can instead write this as a primitive function of the number of friendships formed or $C(s_i, d_i) = c \max[s_i/q_i, d_i/(1 - q_i)]$. Given that all friendships are formed, this it follows that $s_i = t_i q_i$ and $d_i = t_i(1 - q_i)$, and so this is the same cost as that in the given function.

EXAMPLE 1 *An Illustration of the Determination of Stocks and Flows*

There are two types and at every time step 60% agents of type i ($w_i = \frac{3}{5}$) and 40% agents of type j enter the process ($w_j = \frac{2}{5}$).

Suppose that type i 's agents represent $q_i = \frac{2}{3}$ of those in the matching process, so that type j 's agents are the remaining $q_j = 1 - q_i = \frac{1}{3}$. Then type i is over-represented in the matching process (since $\frac{2}{3} > \frac{3}{5}$), while the opposite is true for type j .

Suppose that the optimal strategy for type i agents, which depends on q_i , is to search for $t_i = 12$ total friends, and then exit the process; while the optimal one for type j 's agents, which depends on $q_j = 1 - q_i$, is to stop after only $t_j = 9$ friends are found. This means that type i agents will participate 12 times in the matching process, while type j 's agents only 9. Deriving q_i comes from $q_i = \frac{12w_i}{12w_i + 9w_j} = \frac{2}{3}$, and then $q_j = 1 - q_i = \frac{1}{3}$. Lemma 1 illustrates how optimal stopping strategies are derived from the utility function U .

Type i agents find an expected number of $s_i = q_i t_i = 8$ same-type friends and $d_i = (1 - q_i)t_i = 4$ different-type friends. Similarly type j agents expect to make $s_j = q_j t_j = 3$ same-type friends and $d_j = (1 - q_j)t_j = 6$ different-type friends.

A first check on the coherence of the system is to see if the cross friendships balance: there are only two types and the expected rate at which all the friendships from i to j are formed must equal the expected rate of those from j to i . This requires that $N_i d_i = N_j d_j$, which is satisfied in our case. ■

In Example 1 the optimal t_i 's in response to the q_i 's are given, without investigating the structure of U that justifies the behavior. In the following we characterize the steady-state equilibria when the only exogenous variables are the N_i 's and the functional form of U .

5.1 Equilibrium

Given flows of agents of different types N_1, \dots, N_n , a *steady-state equilibrium* is a collection of times in the matching, stocks, and relative meeting probabilities, (t_i, M_i, q_i) , for each type i (along with the implied (s_i, d_i) 's and resulting utilities) such that:

- (i) t_i solves (2),
- (ii) $M_i = N_i t_i$, and
- (iii) $q_i = \frac{M_i}{M}$,

where $M = \sum_j M_j$.

(i) requires that the time that a type i agent spends in the matching process be optimal given the matching process.

(ii) requires that the stock of agents of type i in the matching process be determined by the strategies and the inflows.

(iii) requires that the proportions that the agents face in the matching be given by the relative stocks.

We do not add an explicit requirement that the inflows match the outflows, as this is implied by (ii), since the flow of agents of type i who are exiting at any given time will be M_i/t_i , and so the exiting amount will be N_i under (ii).

Note that in equilibrium q_i coincides with the homophily index H_i defined in Section 2, since $s_i = t_i q_i$ and $s_i + d_i = t_i q_i + t_i(1 - q_i) = t_i$, and so $H_i = s_i/(s_i + d_i) = t_i q_i/t_i = q_i$.

The definition of equilibrium is well-specified, and in Appendix A we show how it can be derived via an explicit limit of discrete matching processes.

We maintain the assumption that $U_s(0, 0) > c$ and $U_d(0, 0) > c$, so that agents will enter the matching process.

6 Two Types

In this section we focus on the case in which there are only two types. This provides much of the basic intuitions and insights, making the analysis more transparent. There are some complications when we move to more types that we explore in more detail in Section 7.

6.1 Relative Homophily and Equilibrium Conditions

We begin by showing the equilibrium conditions, in conjunction with the minor conditions of diminishing returns to friendships, imply relative homophily.

PROPOSITION 1 (Relative Homophily) *There exists an equilibrium, and in all equilibria both types are active. Moreover, if $N_i > N_j$ then there exists an equilibrium where $q_i > 1/2$, $d_i < d_j$ and $s_i > s_j$, and so relative homophily is satisfied.*

To see the intuition behind Proposition 1, first note that in equilibrium the numbers of cross friendships formed by each population must be the same. That is, each cross-type

friendship must involve an agent of each type and so overall it must be that $N_i d_i = N_j d_j$.²⁰ Then, the conclusion that $d_i < d_j$ is implied by the fact that $N_i > N_j$. The conclusion that $s_i > s_j$ requires the diminishing marginal utility condition, as if this did not hold then j types would be forming more friendships of both types, which would lead to lower overall marginal utility, and both types' marginal utility from staying in the process has to be exactly equal to the cost of a matching, c , at their point of exit. Finally, note that, as discussed in Section 3, if $d_i < d_j$ and $s_i > s_j$, then the homophily indices are also ordered: $H_i = \frac{s_i}{s_i + d_i} > H_j = \frac{s_j}{s_j + d_j}$ implying relative homophily.

The proof of Proposition 1 (in Appendix B, as all the following proofs) uses the properties of the steady state equilibrium and does not require any structure of preferences other than diminishing returns.

We now show that without any bias in preferences, one could not observe the other stylized facts. That is, if preferences are not sensitive to the ratio of same type friendships to different types, then there will be no inbreeding homophily and all types would form the same number of friendships. This is important to note, because it tells us that a bias in preferences will be necessary in order to match the observed data. In particular, the next proposition states the result that when preferences do not pay any attention to types (in the sense of note 21 below), then the mix of friendship types generated in steady state reflects relative population sizes. While this seems fairly obvious, it turns out not to hold with more than two types and thus is slightly more subtle than it might appear.

PROPOSITION 2 (Baseline Homophily) *Let U be symmetric²¹ and let there be two types. Then the unique steady-state equilibrium satisfies baseline homophily; moreover, all types form the same numbers of friendships.*

The fact that preferences are symmetric implies that the two groups must spend the same amount of time in the matching, and then this implies that the relative stocks are the same as the flows, and so friendships are formed in ratios reflecting the inflows.

Baseline homophily is not particular to symmetric preferences, but can also come out of other forms of preferences. This is illustrated in the following example.

EXAMPLE 2 *Log-utilities*

²⁰To see where this is implied in the equilibrium conditions (i)-(iii), note that by (ii) and (iii), it would have to be that $q_1 = N_1 t_1 / (N_1 t_1 + N_2 t_2)$, which implies that $N_2 t_2 q_1 = N_1 t_1 (1 - q_1)$. Then, since $q_1 = 1 - q_2$, this implies that $N_2 t_2 (1 - q_2) = N_1 t_1 (1 - q_1)$ or $N_i d_i = N_j d_j$.

²¹ U is **symmetric** if $U(s, d) = U(d, s)$ for all s and d .

Let $U(s, d) = \alpha \log s + \beta \log(s + d) + \gamma \log d$, where α , β and γ are nonnegative. Equation (3), implies that for any type i , the optimal number of friends is $t_i = \frac{s_i}{q_i} = \frac{\alpha + \beta + \gamma}{c}$, independently of q_i .

Condition (iii) of steady-state equilibrium implies that for any type i :

$$q_i = \frac{N_i t_i}{\sum_j^K N_j t_j} = \frac{N_i}{\sum_j^K N_j} = w_i \quad .$$

The steady state equilibrium is then easy to compute, for any type i : $s_i = q_i t_i = w_i \frac{\alpha + \beta + \gamma}{c}$ and $d_i = (1 - q_i) t_i = (1 - w_i) \frac{\alpha + \beta + \gamma}{c}$. ■

6.2 Same Type Bias, Numbers of Friends and Homophilous Behavior

Proposition 2 shows that although equilibrium conditions and some minimal diminishing returns conditions imply relative homophily, the empirical evidence of inbreeding homophily and increasing numbers of friends with group size require additional structure on preferences. Deriving inbreeding homophily or additional numbers of friends by group size, must then involve bias in preferences between same and different types of friends, as we can deduce from Proposition 2.

U exhibits **same-type bias** if for all $s > d$ and $a > 1$

$$U(as, ad) - U(s, d) > U(ad, as) - U(d, s). \quad (4)$$

This condition requires that the marginal gains from scaling up the number of friends is greater when the mix of friends is biased in favor of own type.

PROPOSITION 3 (Inbreeding Homophily and Increasing Numbers of Friends) *Let preferences satisfy same-type bias and the inflow of types i 's be larger than that of type j 's, so that $N_i > N_j$. Then there exists an equilibrium such that the larger group forms more total friendships per capita than the smaller group (that is, $s_i + d_i > s_j + d_j$). Moreover, the equilibrium exhibits inbreeding homophily for the larger group i and inbreeding heterophily for the smaller group j , and relative homophily is satisfied.*

The proof is based on the fact that, under same-type bias, facing more same types in the matching (a higher q_i) implies higher marginal returns from friendships. This implies that the larger group will spend more time in the matching process, which implies that they form more total friendships. This biases the stock of the larger group in the matching process to

be larger than their relative flow. So, $q_i > w_i$ for the larger group. This leads to inbreeding homophily for the larger group and heterophily for the smaller group. This proposition thus points out that the positive relation between group size and total number of friends, and the tendency (in particular for majorities) to display inbreeding homophily, can be derived from a same-type bias in friend.

Inbreeding homophily here is driven by preferences and choice, but ends up exhibiting opportunity elements as well. More precisely, biased preferences are responsible for larger groups staying longer in the matching, which in turns introduces a bias in the meeting process towards larger groups, and so on. In other terms, the inbreeding homophily of larger groups is generated by an equilibrium feedback between choice and opportunities.

We conclude this section with an example of the steady-state outcome for a simple form of utilities that, for some parameters, satisfy same-type bias.

EXAMPLE 3 *Power α utilities.*

$U(s, d) = (s + \gamma d)^\alpha$, with $\alpha \in (0, 1)$ and $\gamma > 0$. (3) implies that the optimal number of friends for a type i agent is

$$t_i = \left(\frac{\alpha}{c}\right)^{\frac{1}{1-\alpha}} \left((1-\gamma)q + \gamma\right)^{\frac{\alpha}{1-\alpha}} .$$

If $\gamma < 1$ then same-type bias is satisfied and t_i is increasing in q , if $\gamma = 1$ then t_i is independent on q_i as in Example 2. If $\gamma > 1$ then t_i is decreasing in q_i .

When $\alpha = \frac{1}{2}$ we are able to compute the steady state equilibria. From condition (iii) of the steady state equilibrium it follows that

$$\frac{q_i}{q_j} = \frac{w_i t_i}{w_j t_j} .$$

Substituting it follows that (calling $q_i = q$, $q_j = 1 - q$ and similarly for w_i and w_j):

$$\frac{q[(1-\gamma)(1-q) + \gamma]}{(1-q)[(1-\gamma)q + \gamma]} = \frac{w}{1-w} .$$

As long as $w \in [0, 1]$, the last equation has only one solution q such that $q \in [0, 1]$, which is:

$$q = \frac{(1-2\gamma)\frac{w}{1-w} - 1 + \sqrt{\left[\frac{w}{1-w} - (1-2\gamma^2)\right]^2 + 4\gamma^2(1-\gamma^2)}}{2\left(\frac{w}{1-w} - 1\right)(1-\gamma)} . \quad (5)$$

Equation (5) is defined, at the limit, also for $w = 0$, $w = \frac{1}{2}$ and $w = 1$, and the result are respectively $q = 0$, $q = \frac{1}{2}$ and $q = 1$, as expected. Figure 6 shows how q depends on w , when $\gamma < 1$ and $\gamma > 1$. ■

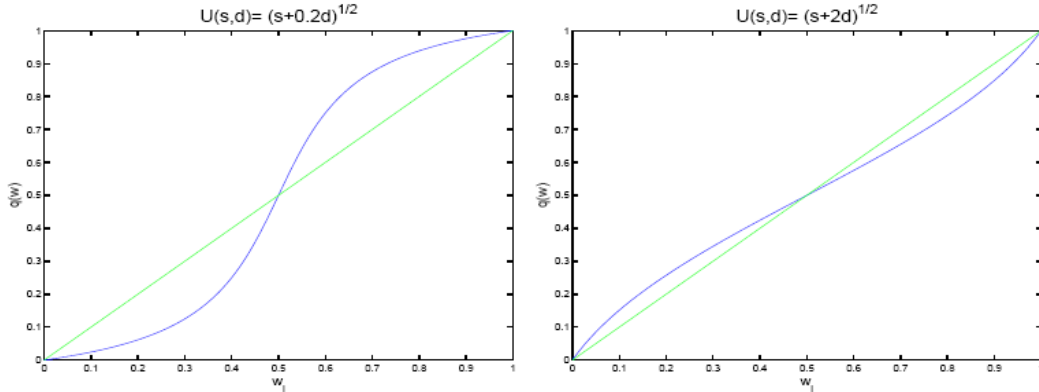


Figure 6: The behavior of q as a function of w when $U(s, d) = \sqrt{s + \gamma d}$. On the left hand side $\gamma = \frac{1}{5}$, and on the right hand side $\gamma = 2$. The green line is the identity.

6.3 Inbreeding Homophily and the Matching Process

Proposition 3 generates the observed relation between group size and total number of friends, and exhibits inbreeding homophily for the larger group. However, it also predicts that the smaller group *necessarily* should be heterophilous, which is not always consistent with the empirical evidence discussed in Section 3, where we saw inbreeding homophily for most groups, and moreover largest inbreeding homophily for middle-sized groups.²²

To understand why, consider the measure of inbreeding homophily

$$IH_i = \frac{q_i - w_i}{1 - w_i} = \frac{\frac{s_i}{t_i} - w_i}{1 - w_i}$$

Without any bias in the meetings, $q_1 = 1 - q_2$ (when there are two groups). Therefore,

$$IH_2 = \frac{1 - q_1 - (1 - w_1)}{1 - (1 - w_1)} = \frac{w_1 - q_1}{w_1} \quad (6)$$

Thus, $IH_2 < 0$ if and only if $IH_1 > 0$. In order to have one group have inbreeding homophily, the other group must be heterophilous. This is clearly contradicted by the data exhibited in Figure 5, where there is inbreeding homophily by almost all groups.

This shows that the observed pattern of inbreeding homophily for all groups cannot be accounted for by preferences alone. To obtain the observed patterns we must also introduce some bias in the matching process. A simple modification of our model results in equilibrium

²²When there are only two types, and they are such that $N_i = N_j$, then our model exhibits, by symmetry, $q_i = q_j$ and baseline homophily for both groups.

behavior that matches the observed patterns in the data. The change is in the meeting process. Note that individuals who have homophilous preferences would tend to gain by biasing their search so that it yields high ratios of own type compared to other types.²³ This could be done in various ways, including meeting friends through friends, as well as joining clubs or taking part in activities that are biased towards own type.²⁴

To get an impression of how this might work, let us examine the case of two groups, but let us allow $q_1 + q_2 > 1$. It must still be that, for both groups, q_i represents the fraction of individuals that i meets who are of own type, and $1 - q_i$ the fraction of different type. Also, we still have equilibrium conditions that cross-group friendships must add up, which is condition $w_1 d_1 = w_2 d_2$. This is equivalently written as

$$w_1(1 - q_1)t_1 = w_2(1 - q_2)t_2 \quad .$$

However, this loosens up the analysis so that it is possible to have own-bias in the meeting technology, relative to what the relative stocks are in the matching process.

To do this systematically, we loosen the meeting technology as in a form of directed search, in the following manner. We require that

$$q_1^\beta + q_2^\beta = 1. \tag{7}$$

We correspondingly rewrite our definition of equilibrium to be

- (i) t_i solves (2),
- (ii) $M_i = N_i t_i$, and
- (iii') $q_1^\beta + q_2^\beta = 1$,
- (iv) $M_1(1 - q_1) = M_2(1 - q_2)$.

When we defined equilibrium previously, (iv) was implied by (ii) and (iii), as discussed after Proposition 1. Under the new matching technology (7), (iii) no longer applies but is replaced by the biased meetings, and so instead we impose (iv) directly.²⁵

²³Such biases in matching processes have appeared in the search literature under the title “directed search”, as in Acemoglu and Shimer (1999).

²⁴Here we are therefore considering a different bias in opportunities from the one driven by choice and endogenous to the model, discussed after Proposition 3. The inbreeding homophily we obtain here contains therefore elements of choice, of choice driven bias in opportunities and of “technological” bias of opportunities.

²⁵Note that (iii') and (iv) imply (iii) when $\beta = 1$.

When $\beta = 1$, this is the uniformly random meeting technology that we have examined before.²⁶ However, when $\beta > 1$, this leads to higher q_i than under uniform meeting, and actually leads to the highest impact for the smallest groups. Note that these groups have the most incentive to meet their own types, as they are not meeting them naturally. What is important here is that the actual q_i will be equal to $f(M_i/M)$ for some function f which is the same for all groups. In other words, we are introduce a change in the meeting process which is the same for all groups.

Note that even though Figure 5 shows the greatest homophily for middle-sized groups, that does not mean that the bias in meeting has to be greatest for them. In fact, what we saw from the model without any meeting bias was that small groups would end up with negative homophily measures. Thus, in order to match the observed data small groups need to have the largest bias in meetings as the equilibrium balance conditions without any bias would force them to have quite negative homophily measures. This is in line with the β exponent in (7).

For instance, when $w_1 \geq w_2$ and t is an increasing function, then (iv) has a solution where q_1 is a continuous function of q_2 , with $1 > q_1 > 0$ for $q_2 = 0$ and $q_1 = 1$ for $q_2 = 1$. As this is a continuous function, it will intersect with the line determined by (iii') and there exists an equilibrium. We use this to solve the following example.

EXAMPLE 4 *An Example with Inbreeding Homophily for All Agents*

Reconsider the utility function from Example 3:

$$U(s, d) = (s + \gamma d)^\alpha = (tq + \gamma t(1 - q))^\alpha$$

for some $\gamma > 0$. Here, when $c = 1$, we satisfy (i) when

$$t(q) = (\alpha(1 - \gamma)q + \alpha\gamma)^{\frac{1}{1-\alpha}}.$$

If we then set $q_2 = (1 - q_1^\beta)^{\frac{1}{\beta}}$, it solves (iii'), we substitute that into (iv) and solve for q_1 , where we substitute for the M_i 's from (ii). In a case where $\alpha = .5$ and $\beta = 2$, these can be solved explicitly; more generally we can solve these numerically and see how the equilibrium varies with the size of the larger group w_1 .

²⁶The formal derivation appearing in Appendix A, which justifies our model as a limit of well defined meeting processes, is based on a completely uniform random mixing process as in Alós-Ferrer (1999). Extending that justification for the processes here requires extending the Alós-Ferrer (1999) results to a more general set of processes where meeting probabilities are biased. This appears to work, but working that out is a project in itself that would take us well beyond the scope of this paper. For the purposes of this section, it is enough to verify that it works for the examples in question, which one can do directly.

A typical figure of the inbreeding homophily is shown in Figure 7. Here $\alpha = .5$, $\beta = \frac{5}{3}$ and $\gamma = .9$. The picture shows the behavior of the inbreeding homophily index $IH_i = \frac{q(w_i) - w_i}{1 - w_i}$, whose shape is not far from the scatter in Figure 5. ■

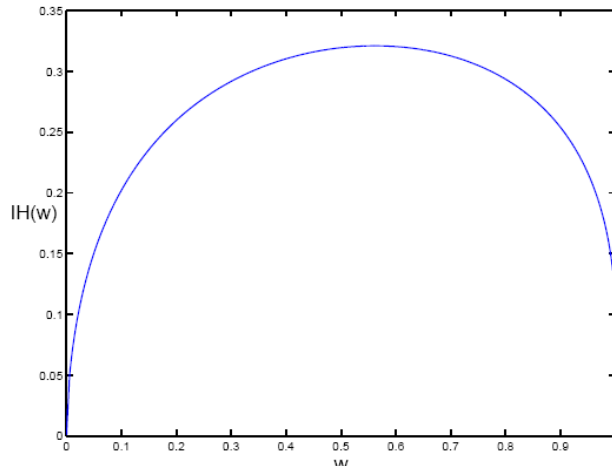


Figure 7: Inbreeding Homophily by Relative Group Size in Example 4 with $\alpha = .5$, $\beta = \frac{5}{3}$ and $\gamma = .9$ ($c = 1$).

7 Many types

In this section we show that passing from two to many types is not trivial, and that some of the above results do not carry over, with some interesting and unexpected new insights. In particular, we show that: *i*) relative homophily can fail among some groups even under same-type biased preferences, *ii*) baseline homophily may not be satisfied even when preferences are perfectly neutral with respect to types; but *iii*) the positive relation between total number of friends and group size carries over, as does the positive relation between inbreeding homophily and group size.

These results are shown in Proposition 4, which exploits a condition on preference similar to “same-type bias”.

U satisfies **strong same-type bias** if

$$U(ax, ay) - U(x, y) \geq U(az, aw) - U(z, w) \tag{8}$$

for any x, y, z, w and a such that $x + y = z + w$, $z < w$, $w \geq x \geq z$, and $a \geq 1$, where (8)

holds with strict equality when $w > x > z$.²⁷

Intuitively, the above condition requires that in the (s, d) -space, the ray along which marginal utility of a new friend is maximal is on or above the 45 degree line. Notably, the required condition is consistent with perfectly symmetric preferences²⁸ (as a consequence, an equivalent result to Proposition 2 fails).

This provides an analog to Proposition 3 for the case of more than two types.

PROPOSITION 4 *Let there be more than two types and U satisfy strong same-type bias. Then there exists a steady-state equilibrium such that $N_i > N_j$ implies $t_i > t_j$ and $q_i > q_j$ (larger groups make more friends). The largest group displays inbreeding homophily, while the smallest group displays inbreeding heterophily. Moreover, if $N_i > N_j$ then $IH_i > IH_j$ (larger groups display larger inbreeding homophily).*

It is useful to outline the proof. The fact that in equilibrium a higher q implies a higher total number of friends, $t(q)$, follows the same steps as in the proof of Proposition 3, using strong same-type bias. The steady-state condition (iii) then implies that the term $\frac{tw}{q}$ is constant across types. Therefore, the terms $t(q)$ and $\frac{q}{w}$ move together across types. Interestingly, in equilibrium the effect of group size on total number of friends and on inbreeding homophily balance across groups. The fixed point argument used in the proof shows that there exist a steady-state equilibrium in which w and q are positively correlated, from which the conclusion that larger groups display larger inbreeding homophily follows.

The result of Proposition 4 can be intuitively explained as follows. Diminishing overall returns to friendship and strong same-type bias imply that larger groups form more friends in equilibrium. By staying longer in the search process, members of larger groups are found more often than members of smaller groups, which implies a larger index of inbreeding homophily. Note that this equilibrium, feedback between opportunity and preference is at work even when preferences are perfectly symmetric as in note 28. In this case, larger groups move along a friendships mix which is closer to the (symmetric) desired one, and stay longer in the matching pool.

Note that the theoretical result is not in contrast with the bell-shaped plot of the homophily index IH_i versus w_i in Figure 5. Proposition 4 is about a single school and does not

²⁷If the inequality in (8) were strict also for $x = w$, then strong same-type bias (8) would imply same-type bias (4). By allowing for equality when $x = w$ we include symmetric preferences, as discussed in note 28.

²⁸An example of symmetric preference with strong same type bias is $U(s, d) = s^\alpha + d^\alpha$, it can be checked, using (3), that $t(q)$ reaches its maximum for $q = \frac{1}{2}$. All the utilities for which s and d are perfect substitutes, as the one from Example 3, do not exhibit the strict inequality required by taste homophily.

make comparisons across schools. The problem is however that the minority group should always display inbreeding heterophily, while this happens only in 38 out of the 84 schools under consideration. An approach similar to the one used in the end of previous section could correct the model for a better matching with the data.

In Appendix C we provide conditions for which all the steady state equilibria have the properties in Proposition 4.

8 Welfare

The model we have developed can be used to explore welfare issues. The preferences are defined only on one's own friendships, and do not depend on opinions of ideological stands on how society should be organized. On the contrary, most of the sociological literature on this topic has implicitly or explicitly adopted the view that an even mix or maximal diversity is desirable per se (see, for instance, the influential paper by Moody (2001)).

We emphasize that our model misses many important aspects of the benefits of diversity, especially in a learning environment, as mixtures only enter preferences in direct friendships. We also emphasize that it does not directly account for the behavioral influences, diffusion, and other implications of social structure. Nonetheless, the welfare analysis here is important because it points out that the equilibrium effects are such that small changes in preference specifications lead to large differences in welfare conclusions. This suggests that more exacting analyses are needed to fully understand optimal policies in programs that might affect, for instance, racial mixes in schools or the extent to which there is tracking and other aspects within school that constrain the meeting process.

We examine how overall average utility varies as the racial composition of a school is varied. Our formal analysis is restricted to the class of homogeneous preferences, for which an explicit solution for the steady-state equilibrium is possible.

We say that U is homogeneous of degree $\alpha \in (0, 1)$ if $U(ks, kd) = k^\alpha U(s, d)$ for all (s, d) .

PROPOSITION 5 *If U is homogeneous of degree $\alpha \in (0, 1)$, then the steady state equilibrium average utility of group i , $U(s_i, d_i) = U(q_i t_i, (1 - q_i) t_i)$, is proportional to $t_i(q_i)$, the optimal total number of friends of type i . Thus we have $\arg \max_q t(q) = \arg \max_q U(qt, (1 - q)t)$.*

PROPOSITION 6 *If U is homogeneous of degree $\alpha \in (0, 1)$, then, average total welfare $\sum_i w_i U_i$ equates $\frac{w_i}{q_i} U_i$ for all $i = 1, \dots, K$. Therefore, maximizing average welfare corresponds to maximizing $\frac{w_i}{q_i} U_i$ which is equal to $\frac{c}{\alpha} t_i \cdot \frac{w_i}{q_i}$.*

Propositions 5 and 6 show that, with homogeneous preferences, aggregate welfare is proportional to the number of friends t_i formed in equilibrium divided by the term q_i/w_i (the obtained ratio is constant for all types i). This result has an interesting interpretation in terms of the empirical observations concerning numbers of friends and group size, and concerning inbreeding homophily. Both the optimal total number of friends t_i and the q_i/w_i increase with the group size w_i (Proposition 4).²⁹ The change in aggregate welfare triggered by a change in relative populations is the result of these two effects. If the population of type i has increased its weight, the increase in total friends formed by i -type agents has a positive effect, while the increase of homophily of i -type agents has the effect of inhibiting the total friendships made by the other types, with a negative effect on welfare. It is notable that the total effect on welfare of changing the size of a group is captured by a reduced form such as the term $t_i \cdot \frac{w_i}{q_i}$, which clearly highlights the trade-off between these two effects on behavior of increased size.

Consider a society with two groups, i and j , of equal size. The effect of increasing the size of group i (locally) depends on the relative magnitudes of the derivatives of t_i and of $\frac{q_i}{w_i}$ with respect to w_i . If the increase in the number of friends outweighs the increase in homophilous behavior, the total welfare increases. If the effect on total friends eventually mitigates and gets outweighed by the homophily effect, then maximal total welfare will be reached at some configuration with $w_i \in (\frac{1}{2}, 1)$.

Another way of looking at the effect of changing relative populations is to study the change in the total welfare: $w_1U_1 + w_2U_2$. If preferences are homogeneous, this change is proportional to $w_1t(q_1(w_1)) + (1 - w_1)t(1 - q_1(w_1))$, where $q(w_i)$ is the probability derived in the matching process from the percentage w_i . Taking the derivative with respect to w_1 yields

$$t(q_1(w_1)) - t(1 - q_1(w_1)) + w_1t'(q_1(w_1))q' - (1 - w_1)t'(1 - q_1(w_1))q'$$

So there is a direct effect of trading agents at the lower utility for those at the higher one, and then indirect effects of changing the higher utility and the lower utility.

Turning to specific functional forms for the utility function allow us to illustrate this trade-off and how it relates to homophily.

The utility function studied in Example 3 can be used to illustrate how different preferences can lead to opposite policy recommendations. In particular, when preferences are homophilous ($\gamma < 1$) we obtain complete segregation, while heterophilous preferences ($\gamma > 1$)

²⁹Proposition 4 is stated in terms of IH_i rather than q_i/w_i , but the results are proven for both.

yield an optimal solution with two equally numerous groups, the case $\gamma = 1$ is the one of perfect substitutes considered in note 28.

What drives these extreme conclusions is the fact that, although preferences are non satiated in both s and d , still same-type and different-type friends are perfect substitutes. In other words, these preferences lack any taste for diversity, and the fact that the marginal rate of substitution is constant drives the corner solutions in terms of integration and segregation. Once this taste for diversity is introduced, complete segregation may not be an optimal policy any more, even in the presence of strongly homophilous preferences.

If instead, there is some preference for diversity, so that the marginal utility of different-type friends grows enough when d tends to zero, the the welfare conclusions change. One way of doing this is to assume that U satisfies Inada conditions on both s and d separately. A separable function of the type $U(s, d) = h(s) + \gamma h(d)$, with $\lim_{x \rightarrow 0} h'(x) = \infty$ satisfies this requirement, and allows for explicit solutions in the steady state. Consider, for instance, the case where $h(x) = \sqrt{x}$.

This corresponds to $U(s, d) = \sqrt{s} + \gamma\sqrt{d}$. Here the optimal t is:

$$t(q) = \frac{(1 - \gamma^2)q + \gamma^2 + \sqrt{\gamma^2 q(1 - q)}}{4c^2}.$$

It can be checked that $t(q)$ has a maximum at $q = \frac{1}{1+\gamma^2}$. If $0 < \gamma < 1$ then $\frac{1}{2} < \frac{1}{1+\gamma^2} < 1$. Figure 8 shows (numerical approximation) results for total welfare for the case of two types. Even when same-type bias is present, but not too strong (for values of γ above $\frac{1}{2}$) an equal split of the population gives the optimal welfare. For higher homophily (lower γ) the optimal welfare is reached with an unequal distribution, which, however, never reaches full segregation.

In Appendix D we illustrate the problem of organizing more than two groups for the simple case of log-utilities. Although we have seen how these preferences fail to capture some important features of our model, they allow for explicit analytic solutions in the more complex case of many types, and provide some insight on the main trade-offs in that case.

9 Concluding remarks

Our model illustrates how specific observations regarding homophily can be traced to different aspects of matching and friendship formation. In fact, we argue that both choice and chance are necessary to explain the observed data. In particular, a bias in preferences for one's own type is needed to generate differences in per capita friendships, and a bias in opportunities or meetings is needed to generate observed inbreeding homophily patterns. Our

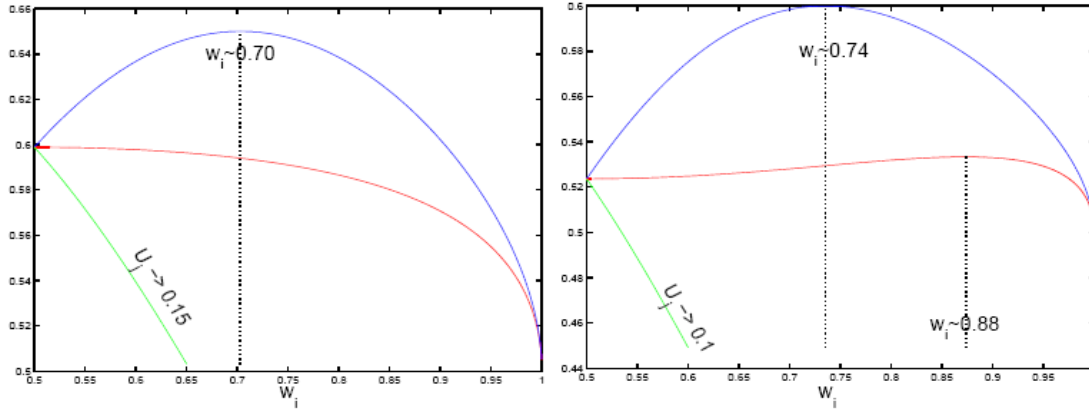


Figure 8: The two plots show the welfare of the two types i (blue) and j (green), and aggregate welfare (red), versus w_i (plotter only from 0.5 to 1, by symmetry), when $U(s, d) = \sqrt{s} + \gamma\sqrt{d}$. The left frame is for $\gamma^2 = 0.3$, where the optimal aggregate welfare is for $w_i = 0.5$. The right hand frame is for $\gamma^2 = 0.2$, where the optimal welfare is for $w_i \simeq 0.88$

analysis also shows the sensitivity of overall welfare to details of the setting. Generally, this suggests that more attention should be paid to modeling the homophily and the patterns of social ties that emerge from variations on matching processes, preferences, and equilibrium conditions, especially given the importance of social structure in many applications. Finally, our analysis suggests that the presence of more than two groups in the economy opens the way to equilibrium phenomena which do not arise when there are only two sizable groups. One of these is that inbreeding homophily may be observed even when preferences are neutral to types, and only as a result of equilibrium restrictions.

The model we have developed here has non satiated preferences, and that is important in some of the conclusions above. In particular, with satiated preferences it can be that agents only desire a limited number of different friendships, and then only accept same-type friends after that (or vice versa). This does not alter the qualitative conclusions above, except for one thing which is that with such preferences it is possible to generate inbreeding homophily for all types without biasing the meeting process. Just as an extreme to illustrate this point, consider a case where agents only derive value from own-type friends and thus refuse all different-type friendships and so all end up with inbreeding homophily of 1. The consideration of satiated preferences complicates the analysis and so we examine it in the Supplementary Material.

We close by mentioning one final caveat for the use of the predictions of our model to

explain aggregate cross-school data. Our theoretical model studies a single matching process (i.e. a single school). In cases where there are just two populations, comparative statics across the size of a group are completely tied down and can be used to make predictions across schools. However, when there are more than two sizeable groups, then the full spectrum of group sizes can be important in determining the outcome for any given group. Although single school statistics are very noisy (given only one to four races in a typical school), some patterns of the type we found in the theoretical analysis seem to be roughly consistent to what we observe within schools. In particular, although only 35% of the schools are exactly ordered by IH_i as they are by w_i , the percentage of non-majorities with a lower inbreeding homophily index than their majorities is 79%.

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Appendix A The Matching Process

In this appendix we provide a foundation for the matching process which justifies our definition of steady-state equilibrium with a continuum of agents. We proceed as follows. First, we show that if agents were to stay an integer number of periods, then there exists a discrete-time random matching process over the continuum so that the process would be in steady-state, and the outgoing agents would have a distribution of realized meetings so that the number of same-types (and hence different-types) that they meet would be governed by a binomial distribution. Next, by taking a limit as the number of periods becomes large, we have a process where the outgoing distribution will converge (in the sense of weak convergence of measures, or convergence in distribution) to a Dirac measure with weight 1 on the expected proportion of meetings with own type, p_i . Finally, we take this limit of large periods by subdividing the matches so that each period an agent makes ε of a new friend, where $\varepsilon \rightarrow 0$, and so many matches are needed to reach t_i total friends.

Consider a discrete-time matching process, with a measure of new types of i entering in each period of N_i and staying an integer T_i number of periods.

This leads to masses of $M_i(T_i) = N_i T_i$ of agents of type i in the matching at any given period and a total mass of $M(T) = \sum_i M_i(T_i)$ of agents in the matching, where $T = (T_1, \dots, T_n)$. Let $p_i(T) = M_i(T_i)/M(T)$ be the proportion of type i 's in the matching.

Agents are labeled from $[0, M(T)]$, with a label a .

The matching-type of an agent a currently in the matching is a vector $\theta_a = (i_a, t_a, p_a)$ which indicates, the type of the agent i_a , how many periods the agent has been in the

matching t_a , and the proportion of same-types that the agent has been matched with to date p_a . Given T , there is a finite set of possible θ 's, denoted by $\Theta(T)$.³⁰

A matching is a Borel measurable bijection $\sigma : M(T) \rightarrow M(T)$, with the property that $\sigma^2(a) = a \neq \sigma(a)$.

A random matching scheme is a probability space with a countably additive probability measure on the space of matchings.

Let μ be a Lebesgue measure and let $I_\theta(a)$ be an indicator function which states whether agent a is of type θ .

Let us begin a process with a fraction of $\nu(\theta)$ of type θ 's.

LEMMA 2 [Alós-Ferrer (1999), Proposition 4.4]³¹ *There exists a matching scheme with probability measure P such that*

P1. $\mu(\sigma(E)) = \mu(E)$ for all Borel E , P -almost surely.

P2. For all a , $P(\sigma(a)$ is of type $\theta) = \nu(\theta)$.

P3. For all θ and θ' ,

$$\int_a I_\theta(a) I_{\theta'}(\sigma(a)) d\mu = \nu(\theta)\nu(\theta').$$

P4. The matching is atomless, so that $P(\sigma(a) = b) = 0$ for all a and b .

Lemma 2 provides a random matching which is measure-preserving (P1), atomless (which can be seen as a minimal form of anonymity) (P4), has the distribution over matches for each agent that is proportional to the relative fraction of matching types in the population (P2), and such that a conclusion equivalent to that of the Law of Large Numbers holds, so that the measure of type i who mix with type j 's is proportional to the product of the proportions in the population (P3).

Thus, Lemma 2 allows for a matching which operates as if the Law of Large Numbers holds for the continuum.

Now, given T , set $\nu((i, t, p)) = \frac{M_i(T)}{T_i} B_{p_i(T), t}(p)$, where $B_{p_i(T), t}(p)$ is the probability of having a fraction of p same types out of t draws with a probability of $p_i(T)$ on each draw, when the matches follow a binomial distribution. This gives the binomial distribution over

³⁰Lemma 4.1 in Alós-Ferrer (1999), given that we start the process with a measurable mapping from agents to matching-types, allows us to relabel agents in a measurable way, so that we can effectively simply partition the interval into subintervals collecting agents of given matching-types together.

³¹See also Duffie and Sun (2007).

possible sequences of realized matches for those of type i who are in their t -th period of matching, who make up a fraction of $\frac{M_i(T)}{T_i}$ of the overall set of type i 's.

Given that we start the types in proportion to what their realized frequency should be under the binomial distribution, it then follows from Lemma 2 (especially P3) that we can find a random matching that gives back the same proportions over types as their new matches and resulting new matching-types will have in a realization that is governed by the binomial distribution.

LEMMA 3 *There exists a random-matching process satisfying (P1)-(P4), such that if ν is as described above, then we are in steady-state so that ν is the outflowing distribution of matching-types.*

What we have shown is that we can find a discrete-time random matching process, so that if each type stayed for an integer number of periods, then we could find a steady-state equilibrium where the outflow of agents of a given type would have a distribution of realized fractions of matchings with same types that matches a binomial distribution.

Next, we take a limit, letting the number of periods become large for each i . We do this as follows. Instead of making a unit of friendship in each period, consider a setting where from each match agents get a ε of friendship, where $\varepsilon > 0$. Then let

$$T_i(\varepsilon) \in \operatorname{argmin}_T |T\varepsilon - t_i|$$

be the number of periods that an agent of type i would have to stay in order to accumulate a total amount t_i of friendships, where t_i is the desired number from the text. For each ε , we have a well-defined $T(\varepsilon)$, and a resulting steady-state matching process where in each period there is an inflow and outflow of a mass of (N_1, \dots, N_n) of the various types of agents, and where the outgoing agents of type i have a distribution over the fraction of same types that they met which is governed by a binomial distribution on draws with $T_i(\varepsilon)$ draws with probability $p_i(T(\varepsilon))$ of meeting own type on each draw. Note that as $\varepsilon \rightarrow 0$, it follows that each $T_i(\varepsilon)\varepsilon \rightarrow t_i$ and that the above described distribution over realized meetings converges (in the sense of weak convergence of measures) to the Dirac measure with mass 1 on p_i . This follows from standard results concerning the limit of a sequence of Binomial distributions (e.g., see Billingsley, 1958).

Appendix B Proofs

Proof of Proposition 1: Without loss of generality, let $N_i \geq N_j$. An equilibrium must specify q_i , M_i , and a t_i for each type such that (i)-(iii) are satisfied. Let $t(q)$ be the function

that assigns the unique optimal t to any $q \in (0, 1)$, as in Lemma 1.³²

As argued following the proposition, it is easy to see that steady-state conditions (ii)-(iii) require that $N_j d_j = N_i d_i$ or $N_j t_j (q_j)(1 - q_j) = N_i t_i (q_i)(1 - q_i)$. Given that $q_j = 1 - q_i$, it follows that a necessary condition for an equilibrium is :

$$\frac{N_j}{N_i} = \frac{t(q_i)(1 - q_i)}{t(1 - q_i)q_i}. \quad (9)$$

We claim that if we find a q_i satisfying (9), then the specification of $t(q_i)$, $M_i = N_i t(q_i)$, q_i together with $t(1 - q_i)$, $M_j = N_j t(1 - q_i)$, $q_j = 1 - q_i$ form an equilibrium. The fact that (i) and (ii) are satisfied follow directly from the definition of t and the way in which the M 's are defined. So let us check that (iii) is also satisfied. Let us verify that $q_i = M_i / (M_i + M_j)$ for each i so that (iii) holds. It is enough to check this for i , given that $q_j = 1 - q_i$. By (9) it follows that

$$N_j t(1 - q_i) q_i = N_i t(q_i)(1 - q_i),$$

or

$$(N_j t(1 - q_i) + N_i t(q_i)) q_i = N_i t(q_i).$$

Then

$$(M_j + M_i) q_i = M_i,$$

which implies the desired conclusion.

So, to establish existence of an equilibrium we show that there exists a q_i satisfying (9). When $q_i = \frac{1}{2}$ it follows that $\frac{t(q_i)}{t(1 - q_i)} = 1$. Thus, if $N_i = N_j$, then $q_i = \frac{1}{2}$ is an equilibrium. Next, note that as q_i increases, the continuous function $\frac{t(q_i)(1 - q_i)}{t(1 - q_i)q_i}$ converges to 0 as $q_i \rightarrow 1$.³³ Thus, when $N_i > N_j$, there always exists a solution to (9) such that $1 > q_i > 1/2$.

The conclusion that $d_i \leq d_j$ with strict inequality when $N_i > N_j$, then follows from the fact that $N_j d_j = N_i d_i$.

Now suppose, contrary to the last part of the proposition, that $s_j \geq s_i$ while $N_i > N_j$. It then follows that $q_j > 1/2$, which is a contradiction since $q_j = 1 - q_i < 1/2$. To see this note that

$$q_j U_s(s_j, d_j) + p_j U_d(s_j, d_j) = c = q_i U_s(s_i, d_i) + p_i U_d(s_i, d_i). \quad (10)$$

Rewrite (10) as

$$q_j (U_s(s_j, d_j) - U_d(s_i, d_i)) = (1 - q_j) (U_s(s_i, d_i) - U_d(s_j, d_j)). \quad (11)$$

³²The uniqueness follows from the conditions on preferences that imply that $q_i U_s(q_i t_i, (1 - q_i) t_i) + (1 - q_i) U_d(q_i t_i, (1 - q_i) t_i)$ is decreasing in t_i .

³³Note that t is bounded above and is bounded away from 0 under our preference conditions.

Since $s_j \geq s_i$ and $d_j > d_i$, the diminishing returns condition and strict concavity of U in each variable implies that

$$U_s(s_j, d_j) + U_d(s_j, d_j) < U_s(s_i, d_i) + U_d(s_i, d_i),$$

which can be rewritten as

$$U_s(s_j, d_j) - U_d(s_i, d_i) < U_s(s_i, d_i) - U_d(s_j, d_j).$$

Thus, for (11) to hold, it must be that $q_j > 1/2$. ■

Proof of Proposition 2: Suppose that $t_i > t_j$. Then for some $a > 1$:

$$t_i = at_j. \tag{12}$$

Equilibrium conditions require:

$$(1 - q_i) [U_s(t_j(1 - q_i), t_j q_i) - U_d(at_j q_i, at_j(1 - q_i))] = q_i [U_s(at_j q_i, at_j(1 - q_i)) - U_d(t_j(1 - q_i), t_j q_i)]. \tag{13}$$

Symmetry implies:

$$U_d(at_j q_i, at_j(1 - q_i)) = U_s(at_j(1 - q_i), at_j q_i). \tag{14}$$

Using (14) we can rewrite (13) as follows:

$$(1 - q_i) [U_s(t_j(1 - q_i), t_j q_i) - U_s(at_j(1 - q_i), at_j q_i)] = q_i [U_d(at_j(1 - q_i), at_j q_i) - U_d(t_j(1 - q_i), t_j q_i)] \tag{15}$$

Inequality (15) clearly violates the assumption of overall diminishing returns to friendships.

Now consider the ratio $\frac{s_i}{d_i}$. We have $N_i d_i = N_j d_j = N_j d_i \frac{q_i}{1 - q_i}$. We obtain $\frac{N_i}{N_j} = \frac{q_i}{1 - q_i} = \frac{s_i}{d_i}$. For type j we have $\frac{N_j}{N_i} = \frac{1 - q_i}{q_i} = \frac{s_j}{d_j}$. ■

Proof of Proposition 3:

Take the equilibrium whose existence is proven in Proposition 1. This equilibrium satisfies relative homophily and is such that $q_i > 1/2$. Consider now condition (4) applied to $(s, d) = (q_i t_j, (1 - q_i) t_j)$; divide now (4) by $a - 1$ and let $a \rightarrow 1$ to obtain:

$$q_i U_s(q_i t_j, (1 - q_i) t_j) + (1 - q_i) U_d(q_i t_j, (1 - q_i) t_j) > (1 - q_i) U_s((1 - q_i) t_j, q_i t_j) + q_i U_d((1 - q_i) t_j, q_i t_j) \tag{16}$$

Optimality conditions require that the RHS of (16) is equal to c . We can therefore conclude that:

$$\frac{\partial}{\partial t} U(q_i t_j, (1 - q_i) t_j) > c. \tag{17}$$

Overall diminishing returns to friendship can now be use to conclude that $s_i + d_i = t_i > s_j + d_j = t_j$ (note that the LHS of (17) is equal to c at the optimal choice t_i for type i).

We therefore obtain

$$t_i = \frac{d_i}{1 - q_i} > \frac{d_j}{q_i} = t_j, \quad (18)$$

from which

$$\frac{q_i}{1 - q_i} > \frac{d_j}{d_i} = \frac{N_i}{N_j} = \frac{w_i}{w_j}. \quad (19)$$

Since $q_i + q_j = 1$ and $w_i + w_j = 1$, (19) becomes $\frac{q_i}{q_j} > \frac{w_i}{w_j}$, from which $q_i > w_i$ (inbreeding homophily) and $q_j < w_j$ (inbreeding heterophily). ■

Proof of Proposition 4:

Let $t(q)$ be the explicit solution to the optimality condition. From (3) we have:

$$\frac{\partial}{\partial t} U(qt, (1 - q)t) = qU_s(qt, (1 - q)t) + (1 - q)U_d(qt, (1 - q)t) = c.$$

Under the properties of U (defined in Section 4, note that they hold also for $s = 0$ or $d = 0$) t is a well defined, continuous and positive single valued function of $q \in [0, 1]$.

We first show that if U exhibits strong same-type bias, then for all $q_j < \frac{1}{2}$ and $q_i \in (q_j, 1 - q_j)$ (since $K > 2$ we have $q_i + q_j < 1$) we have that $t(q_j) < t(q_i)$. Let $z = t(q_j)q_j$, $w = t(q_j)(1 - q_j)$, $x = t(q_j)q_i$, $y = t(q_j)(1 - q_i)$. We take the strict version of same type bias from (8), dividing it by $a - 1$ and letting $a \rightarrow 1$ we obtain:

$$q_i U_s(x, y) + (1 - q_i) U_d(x, y) > (1 - q_j) U_s(z, w) + q_j U_d(z, w) = c, \quad (20)$$

which is just

$$\frac{\partial}{\partial t} U\left(t_j q_i, t_j(1 - q_i)\right) > \frac{\partial}{\partial t} U\left(t_j q_j, t_j(1 - q_j)\right) = c. \quad (21)$$

The assumption of overall diminishing returns to friendship of U implies now that $t(q_i) > t(q_j)$.

We now prove the existence of an equilibrium, satisfying the statement of the proposition, by showing the existence of an interior fixed point of an appropriately constructed function F mapping the $(K - 1)$ -dimensional simplex into itself. F is constructed as follows.

We denote by F_1 the function that associates with each element $\vec{q} = (q_1, q_2, \dots, q_K)$ of the simplex the vector $F^1(q) = (t(q_1), t(q_2), \dots, t(q_K))$.

We then denote by F^2 the continuous function that orders any K -vector in increasing order.

We finally denote by F^3 the continuous function defined as

$$F^3(x_1, x_2, \dots, x_K) = \left(\frac{N_1 x_1}{\sum N_i x_i}, \frac{N_2 x_2}{\sum N_i x_i}, \dots, \frac{N_K x_K}{\sum N_i x_i} \right), \quad (22)$$

where the vector (N_1, N_2, \dots, N_K) is ordered in increasing order.

Note that for any $\vec{q} = (q_1, q_2, \dots, q_K)$, $F(\vec{q}) \equiv F^3 \circ F^2 \circ F^1(\vec{q})$ is ordered in increasing order and the sum of its elements is 1.

The mapping F is a continuous function from the simplex to itself, and possesses therefore a fixed point by Brouwer's Theorem.

Note that, on the fixed point, F^2 preserves the ordering of the vector \vec{q} because $t(q)$ is increasing, so that F^2 applied to the fixed point is just the identity function.

We conclude the proof by showing that such a fixed point is interior. Suppose not, so that there is at least one q_i such that $q_i = 0$. By the assumption that $U_s(0, 0) > c$ and $U_d(0, 0) > c$, $F^3 \circ F^2 \circ F^1$ has all positive values (because F^1 has all positive values), which implies that $q_i = 0$ cannot be an element of the fixed point.

We finally show that, given the steady state condition $\frac{t_i w_i}{q_i} = \frac{t_j w_j}{q_j}$, the vector $(\frac{q_1}{w_1}, \frac{q_2}{w_2}, \dots, \frac{q_K}{w_K})$ is ordered in increasing order (given that the vector (t_1, t_2, \dots, t_K) is also increasing). Given that $\sum q_i = 1$ and $\sum w_i = 1$, we conclude that $\frac{q_K}{w_K} > 1$ while $\frac{q_1}{w_1} < 1$. It is now straightforward to show that this implies, from $N_i > N_j$, that $\frac{q_i - w_i}{1 - w_i} > \frac{q_j - w_j}{1 - w_j}$. ■

Proof of proposition 5: Homogeneity of degree α , together with the condition for the optimal stopping:

$$c = \frac{\partial}{\partial t} U(qt, (1 - q)t)$$

implies that:

$$t(q) = \alpha \frac{U(qt, (1 - q)t)}{c},$$

which implies for each type i that:

$$U_i(q_i t(q_i), (1 - q_i) t(q_i)) = \frac{c t(q_i)}{\alpha}, \quad (23)$$

so that the optimal number of friends given q_i is proportional to the maximized level of utility up to the constant $\frac{\alpha}{c}$. ■

Proof of proposition 6: Equation (iii') for steady state equilibrium and $s_i = q_i t_i$ imply that

$$\overline{M} = \frac{N_i t_i}{q_i}$$

for each i .

This implies that

$$\frac{N_i t_i}{q_i} = \frac{N_j t_j}{q_j}$$

for every i and j .

Therefore, in equilibrium

$$\frac{w_i U_i}{q_i} = \frac{w_j U_j}{q_j}.$$

Maximizing the aggregate (or average) utility in society, then amounts to maximizing

$$\sum_i w_i U_i = w_1 U_1 \left(1 + \sum_{j \geq 2} \frac{q_j}{q_1} \right) = \frac{w_1}{q_1} U_1 = \frac{c w_1}{\alpha q_1} t_1,$$

which establishes the claim. ■

Appendix C Ruling out non well-ordered equilibria

In this section we establish conditions under which the result of Proposition 4 apply to all steady state equilibria. Recall that U is homogeneous of degree $\alpha \in (0, 1)$ if $U(ks, kd) = k^\alpha U(s, d)$ for all (s, d) .

PROPOSITION 7 *Let $U(s, d)$ be homogeneous of degree $\alpha \in (0, \frac{1}{2}]$. In every steady state equilibrium, $N_i > N_j$ implies $q_i > q_j$. If preferences also exhibit same type bias, then $t_i = s_i + d_i > t_j = s_j + d_j$ whenever $N_i > N_j$, and i exhibits more inbreeding homophily than j ($IH_i > IH_j$).³⁴*

Proof of Proposition 7: From condition (iii) of equilibrium we have that if $N_i > N_j$, then $\frac{t_i}{q_i} < \frac{t_j}{q_j}$.

If $\frac{\partial t}{\partial q} < 0$ we have that $q_i > q_j$.

$\frac{\partial t}{\partial q} = \frac{\frac{\partial t}{\partial q} q - t}{q^2} < 0$ if and only if $\frac{\partial t}{\partial q} \cdot \frac{q}{t} < 1$.

Remember (3):

$$\frac{\partial}{\partial t} U(qt, (1-q)t) = c \quad . \quad (24)$$

³⁴The next example shows that, even with two types, if U is homogeneous of degree $\alpha > \frac{1}{2}$, we can have equilibria with $N_i > N_j$ but $q_i < q_j$ and $t_i < t_j$. Consider $U(s, d) = s^{0.9} + \frac{1}{2}d^{0.9}$, $2N_1 = N_2$ and $c = 0.9$. Given homogeneity it is possible to express explicitly

$$t(q) = (q^{0.9} + (1-q)^{0.9})^{10} \quad .$$

We find numerically two steady state equilibria: 1) a non well-ordered equilibrium in which $q_1 \simeq 0.62$, $q_2 \simeq 0.38$, $t_1 \simeq 0.23$ and $t_2 \simeq 0.07$; 2) a well-ordered equilibrium in which $q_1 \simeq 0.001$, $q_2 \simeq 0.999$, $t_1 \simeq 0.0001$ and $t_2 \simeq 1.00$.

considering that U is homogeneous of degree α we get:

$$t(q) = \alpha \frac{U(qt, (1-q)t)}{c} . \quad (25)$$

We can take the derivative of (25) with respect to q , obtaining:

$$\frac{\partial t}{\partial q} = \alpha \frac{t \left(U_s(qt, (1-q)t) - U_d(qt, (1-q)t) \right) + \frac{\partial t}{\partial q} \left(\frac{\partial}{\partial t} U(qt(1-q)t) \right)}{c} , \quad (26)$$

from which, using (24):

$$\frac{\partial t}{\partial q} = \frac{\alpha}{1-\alpha} \frac{t \left(U_s(qt, (1-q)t) - U_d(qt, (1-q)t) \right)}{c} . \quad (27)$$

From (27) we have:

$$\begin{aligned} \frac{\partial t}{\partial q} \cdot \frac{q}{t} &= \frac{\alpha}{1-\alpha} \frac{q \left(U_s(qt, (1-q)t) - U_d(qt, (1-q)t) \right)}{c} \\ &< \frac{\alpha}{1-\alpha} \frac{q U_s(qt, (1-q)t) + (1-q) U_d(qt, (1-q)t)}{c} \\ &= \frac{\alpha}{1-\alpha} \leq 1 , \end{aligned}$$

where the first inequality comes from the fact that $U_d > 0$, while the second comes from the fact that $\alpha \leq \frac{1}{2}$.

The last part of the proof follows from Proposition 4. ■

Appendix D Welfare analysis with many types and logarithmic preferences

We use the logarithmic preferences introduced in Example 2 to investigate alternative policies when there are more than two groups. Note that in this case $U(s, d)$ is not homogeneous any more.³⁵

The functional form $U(s, d) = \alpha \log s + \beta \log(s + d) + \gamma \log d$ exhibits baseline homophily ($q_i = w_i$ for any type i), since the optimal number of friends is always $\frac{\alpha + \beta + \gamma}{c}$, independently on q .

³⁵The \log function can be considered a case of homogeneity α , at the limit $\alpha \rightarrow 0$. It is however easy to check that not all our results concerning homogeneous functions hold in the logarithmic case.

Consider the case in which $\alpha = 1$, $\beta > 0$ and $\gamma = 0$. The utility of type i agents is then:

$$U_i(w_i) = \log \left(w_i \frac{1 + \beta}{c} \right) + \beta \log \left(\frac{1 + \beta}{c} \right) - 1 - \beta \quad ,$$

The utility of type i agents is indifferent to a split of type j in more types, as long as w_i is kept constant. There is no externality on other types if a type splits in more sub-groups. The type i utility is always increasing in w_i . If the number of types is $K = 2$, then the expected average welfare $w_i U(w_i) + (1 - w_i) U(1 - w_i)$ is:

$$w_i \log \left(w_i \frac{1 + \beta}{c} \right) + \beta \log \left(\frac{1 + \beta}{c} \right) + (1 - w_i) \log \left((1 - w_i) \frac{1 + \beta}{c} \right) - 1 - \beta.$$

which is convex, achieves always a minimum at $w_i = \frac{1}{2}$ and a maximum in the completely segregated case where $w_i = 0$ or $w_i = 1$.

Consider now the case in which $\alpha = 1$, $\beta = 0$ and $\gamma > 0$. The utility of type i agents depends on w_i and is:

$$U_i(w_i) = \log \left(w_i \frac{1 + \gamma}{c} \right) + \gamma \log \left((1 - w_i) \frac{1 + \gamma}{c} \right) - 1 - \gamma \quad .$$

There is still no externality if another type is split in more types. It is easy to check that $U_i(w_i)$ is increasing in w_i up to the point where $w_i = \frac{1}{1 + \gamma}$, and then is decreasing. If γ is a positive integer, then the social optimum would be to split equally $\gamma + 1$ types (if available) in the matching process.

If we consider only $K = 2$ types, then the expected average welfare is

$$\left(w_i + \gamma(1 - w_i) \right) \log \left(w_i \frac{1 + \gamma}{c} \right) + \left(w_i \gamma + (1 - w_i) \right) \log \left((1 - w_i) \frac{1 + \gamma}{c} \right) - 1 - \gamma \quad .$$

The previous function has always a critical point in $w_i = \frac{1}{2}$. Its second derivative with respect to w_i is $\frac{-\gamma + (1 + \gamma)w_i(1 - w_i)}{w_i^2(1 - w_i)^2}$, which is negative for any $w_i \in (0, 1)$ only if $\gamma > \frac{1}{3}$. If $\gamma \geq \frac{1}{3}$ then the average welfare obtains its maximum for $w_i = \frac{1}{2}$.

If instead $\gamma < \frac{1}{3}$, then $w_i = \frac{1}{2}$ is a local minimum for the aggregate welfare, whose functional form is symmetric and bimodal, as in Figure 8.

Figure 9 shows the two cases.

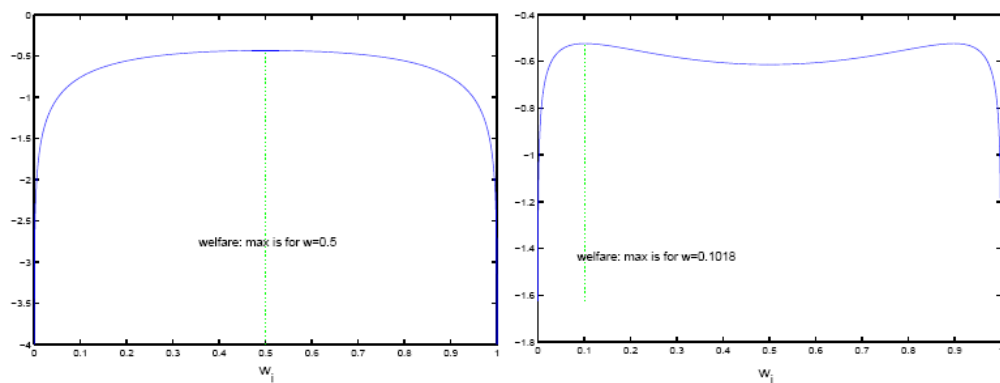


Figure 9: Average aggregate welfare when $U(s, d) = \log s + \gamma \log d$. Left hand side is for $\gamma = \frac{1}{2}$, right hand side is for $\gamma = \frac{1}{5}$.